① W.W. Sawyer Prelude to Mathematics
1st edition, 1955.

Algebra ⇝ Homological Algebra

$\dots -2 \ -1 \ 0 \ 1 \ 2 \ 3 \dots$ $\mathbb{Z}$

rationals $\mathbb{Q}$

reals $\mathbb{R}$ field

complex #s $\mathbb{C}$

Physics ⇝ Linear Algebra

field vector space

k V - v.s. over k

Ex. n $k^n = \{ (x_1, \dots, x_n) \mid x_i \in k \}$

Th $\boxed{\text{Any v. space} \simeq k^n}$ $n := \dim V$

$$\boxed{k^n \xrightarrow[\varphi]{\simeq} V}$$

iso

φ is monic $\ker \varphi = 0$

φ is epic, $\mathrm{Im}\, \varphi = V$.

Next level

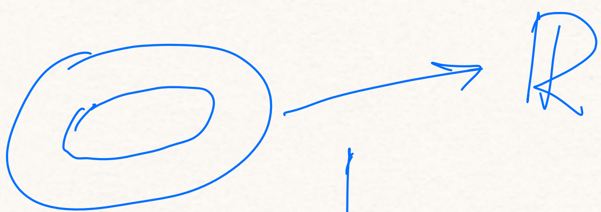
fields $\rightsquigarrow$ ring

vector spaces $\rightsquigarrow$ modules

Ex $\underbrace{\mathbb{Z},\ \cancel{pb}\ \text{poly-s},}_{\text{comm}}$ $\underbrace{\text{Matrices of size } n \times n}_{\text{non-comm}}$

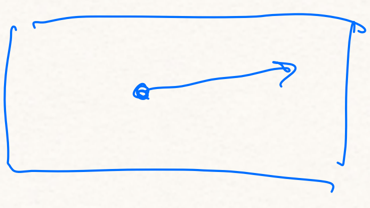
rings of functions | rings of operators

Physics:

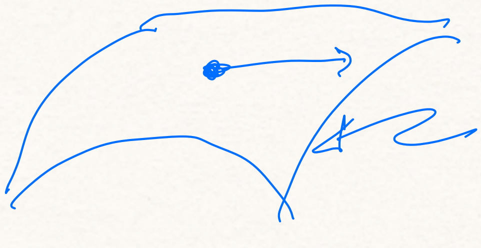


Geometry	Algebra
Object	Functions on object

E.g. quantum group



functions



Important Obs: Rings must be studied.

—"— Rings can be studied via their modules.

Λ $\qquad$ Mod-Λ (Λ-Mod)

---

finite $\qquad$ M module $\not\cong R^n$

$$\begin{tikzcd}
R^{m} \arrow[r, "A"] \arrow[dr, "\text{onto}"'] & R^{n} \arrow[r, "\varphi"] & M \arrow[r] & 0 \\
 & K \arrow[u, "\text{inj}"'] & & \\
0 \arrow[ur] & & &
\end{tikzcd}$$

$\exists$ surjective φ.

A = matrix of size $\boxed{n \times m}$

finitely presented module

$R^m \longrightarrow R^n$

$k^m \longrightarrow k^n$

$\boxed{\text{Modules + Linear Maps}}$

$\rightsquigarrow$

Category of modules

Rings can be studied via their module <u>categories</u>.

Q: How to study categories?

A: Study functors on categories.

$$\mathcal{A} \xrightarrow{F} \mathcal{B}$$

$$\mathrm{Ob}\,\mathcal{A} \longrightarrow \mathrm{Ob}\,\mathcal{B}$$

$$\mathrm{Mor}\,\mathcal{A} \longrightarrow \mathrm{Mor}\,\mathcal{B}$$

<u>Functor categories</u>

Objects = arrows F (functors)

Morphisms =

$$\begin{tikzcd}
F(X) \arrow[r, "F"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi"] \\
G(X) \arrow[r, "G"'] & G(Y)
\end{tikzcd}$$

$$\underline{\text{Top spaces}} \xrightarrow{H_1} \underline{\text{Abelian groups}}$$

Specialize: Cat Λ-Modules

Functors: $\Lambda\text{-Mod} \longrightarrow Ab$

additive

finitely presented

Ex. M $\qquad$ $\Lambda\text{-Mod} \longrightarrow Ab$

$N \longrightarrow (M,N)$

~~Hom~~

representable $\leftarrow (M,-)$

$$\boxed{\begin{array}{c}(N,-) \longrightarrow (M,-) \longrightarrow F \longrightarrow 0 \\ \text{f.p. functor.}\end{array}}$$

Mod Λ $\qquad$ f.p. (Mod Λ, Ab)

$\boxed{w,\ Y^\circ \text{ are adjoint to each other on right}}$

$L^\circ Y^\circ$

X

$$\begin{tikzcd}
\mathrm{Mod}\,\Lambda \arrow[r, bend left=40, "L^\circ Y^\circ"] \arrow[r, bend right=40, "Y^\circ"'] & \mathrm{f.p.}(\mathrm{Mod}\,\Lambda, Ab) \arrow[l, "w = \text{defect}"', "\otimes"]
\end{tikzcd}$$

$M \qquad (M,-)$

$\boxed{L^\circ Y^\circ \text{ and } w \text{ are adjoint to each other on the left}}$

Y° — Yoneda embedding

Q: ~~$\mathbb{Y}$~~ What is the interpretation of ⊗ in computer science?

$$(F, \mathbb{Y}(A)) \simeq (A, w(F))$$

$$\otimes(F, (A,-))$$ co-Yoneda lemma

$$((A,-), F) \simeq F(A)$$

Yoneda's lemma

- $$\boxed{\frac{\text{f.p.}(\text{Mod-}\Lambda, Ab)}{\{F \mid w(F) = 0\}} \simeq \text{Mod-}\Lambda}$$ (Auslander) 1960s

---

$$(B,-) \xrightarrow{(f,-)} (A,-) \to F \to 0$$

$$B \xleftarrow{f} A \leftarrow w(F) \leftarrow 0$$

$$w(F) \simeq \ker f.$$