Mock Theta Functions Background

Ammar Husain

Boston Computation Club
Sep 17, 2022

Outline

Modular Group

Modular Group

Definition (Modular Group)

The modular group is defined as the group $PSL(2,\mathbb{Z})$. $SL(2,\mathbb{Z})$ *describes 2 by 2 matrices with integer entries and determinant* 1 *under the operation of matrix multiplication. The* P *then means we should identify two matrices which differ by a scalar multiple.*

Exercise

Describe what matrices get identified together? Look at the determinant condition.

Presentation

A useful presentation for this group is with two generators S and T. Provide actions for these two satisfying the requisite relations and the rest of the modular group comes along for the ride.

$$
\begin{aligned}
S &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
T &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
S^2 &= -I_2 \equiv I_2 \\
(ST)^3 &= -I_2 \equiv I_2
\end{aligned}
$$

Congruence Subgroups 1

We can impose some sort of constraint on divisibility properties of entries. There are two ways of particular importance.

Definition (Principal Congruence subgroup $\Gamma(N)$)

Those matrices which reduce to the identity under the homomorphism $\pi_N \; SL(2,\mathbb{Z}) \twoheadrightarrow SL(2,\mathbb{Z}/N\mathbb{Z})$ which reduces all entries modulo N. This is a finite index subgroup where the index depends on the primes dividing N.

Definition ($\Gamma_0(N)$)

Only c has to be divisible by N.

$\Gamma(N)$ is manifestly a subgroup of $\Gamma_0(N)$.

Congruence Subgroups 2

In general a congruence group of level N is defined as any subgroup of $SL(2, \mathbb{Z})$ containing $\Gamma(N)$ where N is taken as small as possible. From any subgroup G of the finite group $SL(2, \mathbb{Z}/N\mathbb{Z})$ we can consider $\pi_N^{-1}(G)$ to get a congruence subgroup but it might be of smaller level because even though it contains $\Gamma(N)$ by construction, it could contain a $\Gamma(M)$ as well (use the degenerate case when G is everything so the inverse image is all of $SL(2, \mathbb{Z})$).

$(2, 3, n)$ triangle groups

A triangle group is a group generated by reflections across the sides of a "triangle" with angles $\frac{\pi}{l}$, $\frac{\pi}{m}$ and $\frac{\pi}{n}$. Depending on the sum of these angles whether we are considering Euclidean, spherical or hyperbolic triangles changes. Being reflections we see that the 3 generators all square to the identity. Because the composition of two reflections is a rotations and we know the angles we get relations $(ab)^l = e$ etc where l matches the angle between the sides considered by the reflections a and b. The modular group already is isomorphic to $T(2, 3, \infty)$ (The relation $(ca)^n$ being omitted because infinity means it has no prescribed order)

Upper Half Plane

Definition (H^2)

The subset of the complex plane with imaginary part greater than 0. This is a model space (Poincaré and Klein models too) for hyperbolic plane geometry in the sense that we describe other hyperbolic surfaces as quotients thereof.

2D Lattices

Consider the problem of classifying 2d lattices up to homotethy $z\Lambda \equiv \Lambda$. If a basis $\omega_1, \omega_2 \in \mathbb{C}$ was given, we could scale by ω_1^{-1} so the lattice would be equivalent to the one spanned by 1 and $\frac{\omega_2}{\omega_1}$. The latter is nonreal by virtue of the original 2 not being colinear, but it might be in the lower half plane. If it is in the lower half plane, note that the lattice spanned by 1 and $\frac{\omega_2}{\omega_1}$ is the same as that spanned by 1 and $-\frac{\omega_2}{\omega_1}$.

Modular Action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}$$

Scaling the matrix does not effect the result so this is a well defined action for $PSL(2, \mathbb{Z})$ even though we only wrote it for $SL(2, \mathbb{Z})$.

Exercise

Show the result is still in the upper half plane.

Modular Form

Modular Forms

Definition (Modular Form)

A modular form for the finite index subgroup $\Gamma \subseteq SL(2,\mathbb{Z})$ *and weight* k *is defined as a holomorphic function* f *on the upper half plane subject to the following*
For all $\gamma \in \Gamma$*,*

$$f(\gamma(z)) = (cz+d)^k f(z)$$

$$(cz+d)^{-k} f(\gamma(z))$$

is bounded as $Im(z) \to +\infty$ *for all* $\gamma \in SL(2,\mathbb{Z})$*.*

If we take Γ to be all of $SL(2,\mathbb{Z})$ the statement becomes easier because the bounded condition just states $f(z)$ is bounded in that case.

Fourier Series

If Γ is of weight N, then T^N is in $\Gamma(N)$ and therefore Γ as well. Therefore a modular form for such a group will satisfy
$f(z+N) = f(T^N(z)) = (0z+1)^k f(z) = f(z)$. f is periodic so it has a Fourier series.
Taking Γ to be all of $SL(2, \mathbb{Z})$, we get that f is periodic with period 1. This means we can write

$$f(z) = \sum_{n=-m}^{\infty} a_n e^{2\pi i n z}$$

The index has a lower bound thanks to the fact that $f(z)$ is bounded as the imaginary part grows. However the lower bound $-m$ may be very negative. (m is an integer but the negative sign is not absorbed into it to avoid thinking the lower bound is non-negative)

q-Series

We see that $e^{2\pi i z}$ is common and it is always being raised to integer powers so we can do some rewriting

$$q = e^{2\pi i z}$$

$$f(z) = \sum_{n=-m}^{\infty} a_n q^n$$

Assuming $m > 0$, this has a pole of order m at $q = 0$ or a pole at $i\infty$ in terms of z. This says f is meromorphic at $q = 0$.

Eisenstein Series

$$G_k(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^{2k}}$$
$$G_k(\tau+1) = G_k(\tau)$$
$$G_k(-1/\tau) = \tau^{2k} G_k(\tau)$$

These functions come up as expanding the Weirstrauss function $\wp(z;\tau)$ near $z=0$ which is the simplest doubly-periodic meromorphic function.

j-Invariant

Definition

The j-invariant is defined in terms of Eisenstein series as follows

$$g_2(\tau) = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^4}$$

$$g_3(\tau) = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^6}$$

$$\Delta(\tau) = g_2^3(\tau) - 27g_3^2(\tau) = (2\pi)^{12}\eta^{24}(\tau)$$

$$j(\tau) = 1728\frac{g_2^3(\tau)}{\Delta(\tau)}$$

Knowing g_2 is modular of weight 4 and Δ is modular of weight 12 we see that this would be a modular form of weight 0 except it is meromorphic instead of holomorphic. This is what is known as a modular function.

j-invariant

In fact all modular functions are given by taking rational functions of this j-invariant.
As before q-series still applies.

$$j = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$$

Those are dimensions of a representation of the monster group.

η Function

$$
\begin{aligned}
\eta(\tau) &\equiv e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}) \\
\eta(\tau+1) &= e^{\pi i/12}e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi in}e^{2\pi in\tau}) \\
&= e^{\pi i/12}e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}) \\
&= e^{\pi i/12}\eta(\tau) \\
\eta(-1/\tau) &= (-i\tau)^{12}\eta(\tau)
\end{aligned}
$$

η Function

Those formulas gave the actions of T and S. In general the transformation rule is more complicated.

$$\eta(g\tau) = \epsilon(a,b,c,d)(c\tau + d)^{1/2}\eta(\tau)$$

where ϵ is a certain explicit 24th root of unity which depends on $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

η Function

So this is like a modular form of weight $1/2$ except with a factor of automorphy. But we can just take the appropriate power so those subtleties disappear.

One can get this $1/2$ behavior without explicitness on the ϵ using a Liouville like argument on $\frac{\Delta}{\eta^{24}}$ to get that η^{24} is proportional to the modular form of weight 12 given by Δ.

Jacobi ϑ function

$$
\begin{aligned}
\vartheta(z;\tau) &= \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z} \\
&= \sum q^{n^2} e^{2\pi i n z} \\
\vartheta_{00}(z;\tau) &= \vartheta(z;\tau) \\
\vartheta_{01}(z;\tau) &= \vartheta(z+1/2;\tau) \\
\vartheta_{10}(z;\tau) &= e^{\pi i/4\tau + \pi i z}\vartheta(z+1/2\tau;\tau) \\
\vartheta_{11}(z;\tau) &= e^{\pi i/4\tau + \pi i(z+1/2)}\vartheta(z+1/2+1/2\tau;\tau)
\end{aligned}
$$

We want functions of only τ or equivalently q so set $z=0$.

Jacobi ϑ function

$$
\begin{aligned}
\vartheta_{00}(0;\tau) &= \sum q^{n^2} \\
\vartheta_{01}(0;\tau) &= \sum (-1)^n q^{n^2} \\
\vartheta_{10}(0;\tau) &= \sum q^{(n+1/2)^2} \\
\vartheta_{11}(0;\tau) &= -\sum (-1)^{n-1/2} q^{(n+1/2)^2}
\end{aligned}
$$

Again modular of weight 1/2 with some subtlety about automorphy factor and congruence group.

Integer Partitions

Integer Partitions

Definition ($p(n)$)

$p(n)$ is defined as the number of ways to partition a natural number n as a sum of positive natural numbers. The order of the summands does not matter (in contrast with composition numbers).

- $p(0) = 1$ because only the empty sum works
- $p(1) = 1$ because only $1 = 1$ provides such a partition
- $p(2) = 2$ because we can write both $2 = 2$ and $2 = 1 + 1$

Exercise

Write a function that inputs n and gives an iterator over the partitions of n. The count of items if all are iterated over has to be $p(n)$.

Generating Function

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$
$$= \prod_{k=1}^{\infty} \sum_{l=0}^{\infty} x^{lk}$$

Consider the coefficient of x^n on the RHS. Each contribution to that term is given by picking one value l_k for each k such that $\sum_{k=1}^{\infty} l_k k = n$. The number of ways to do this provides the coefficient. This gives a partition of n by saying there are l_k summands of k being repeated. $l_k = 0$ for all $k > n$ and that all $l_k < n$. You can do the truncations first before manipulations if you feel more comfortable with finite use of the distribution law instead.

Restricted Parts

Suppose we restrict the parts to be in a set R instead of all positive natural numbers.

$$\sum_{n=0}^{\infty} p_R(n)x^n = \prod_{k\in R} \frac{1}{1-x^k}$$

Follow the same reasoning mutatis munandis.

Odd/Distinct Parts

Let R be the set of odd numbers. So we get

$$\sum_{n=0}^{\infty} p_{odd}(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}}$$

Also look at partitions into distinct parts so $l_k = 0, 1$ only.

$$\sum_{n=0}^{\infty} p_{distinct}(n)x^n = \prod_{k=1}^{\infty} (1+x^k)$$

where for each k we choose the 1 if we want no part of size k or the x^k if we do want a part of size k. So counting all these contributions gives the number of ways to break up n as a sum where all the summands are distinct.
These two counts are always the same.

Modularity + Integer Partitions

Modularity + Integer Partitions

$$q^{-1/24}\eta = \prod_{n=1}^{\infty}(1 - q^n)$$

$$q^{1/24}\frac{1}{\eta} = \sum_{n=0}^{\infty} p(n)q^n$$