The Yoneda Lemma

A (mostly) informal discussion of what it is.

Bryson

November 1, 2020

Statement of the Yoneda Lemma

Lemma (Yoneda)

The following is a natural isomorphism in c and X:

$$\begin{array}{ccc} \eta & \longmapsto & \eta_c \operatorname{Id}_c \\ [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]\big(Yc, X\big) & \xleftrightarrow{\cong} & Xc \\ X - x & \longleftarrow\!\!\shortmid & x \end{array} \tag{1.1}$$

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We need a place to do some math.

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Categories provide a general enough framework to get on with.

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Notation: Given a pair of objects $\bullet, \circ$ in a category C, we write $\mathrm{C}(\bullet, \circ)$ meaning "the collection of morphisms from $\bullet$ to $\circ$"

Examples

Set: the category of sets

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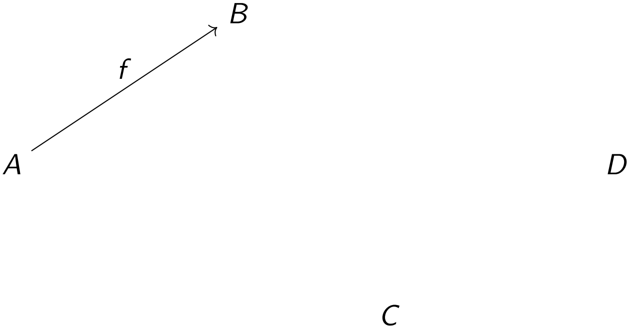

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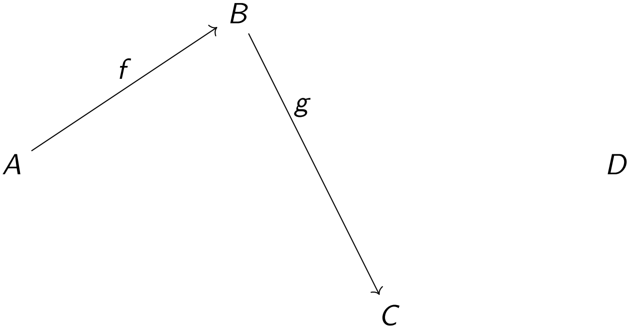

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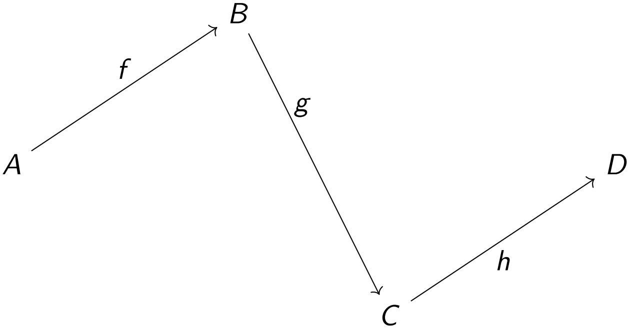

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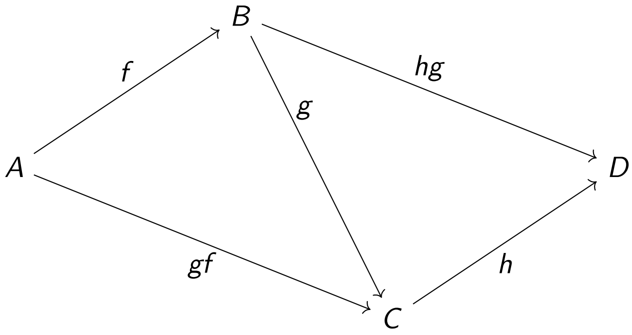

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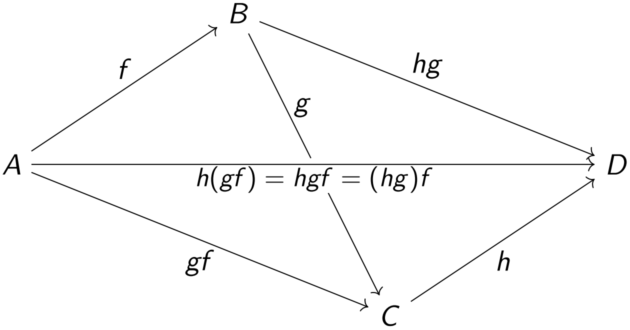
and it's associative



Examples

Top: the category of topological spaces & continuous maps

Grp: the category of groups & group homomorphisms

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C^{op}: the *opposite category* of a given category C, having the same objects but reversing all arrows

The category of paths in a space, whose objects are points and morphisms are continuous paths between points.

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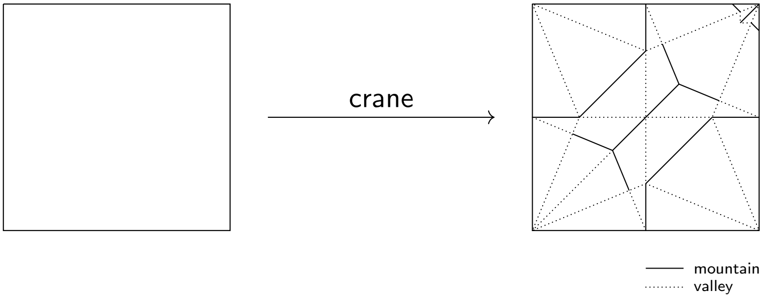
~~The category of paths in a space, whose objects are points and morphisms are continuous paths between points.~~ **Exercise: not associative**

...

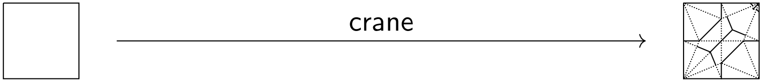
It is very easy to come up with examples of categories.

Define Gami as the category whose morphisms are the action of adding folds to a square sheet of paper.

Here's a pattern for an origami crane:



Breaking it into Steps



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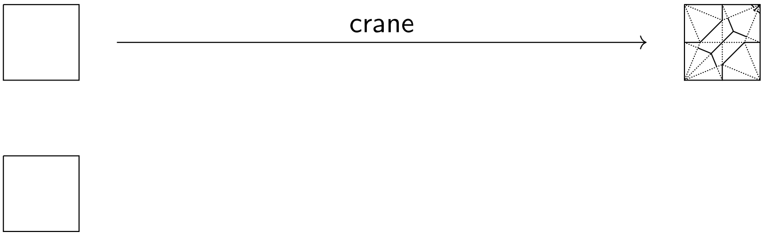

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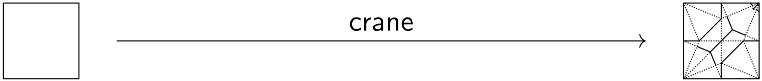

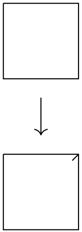

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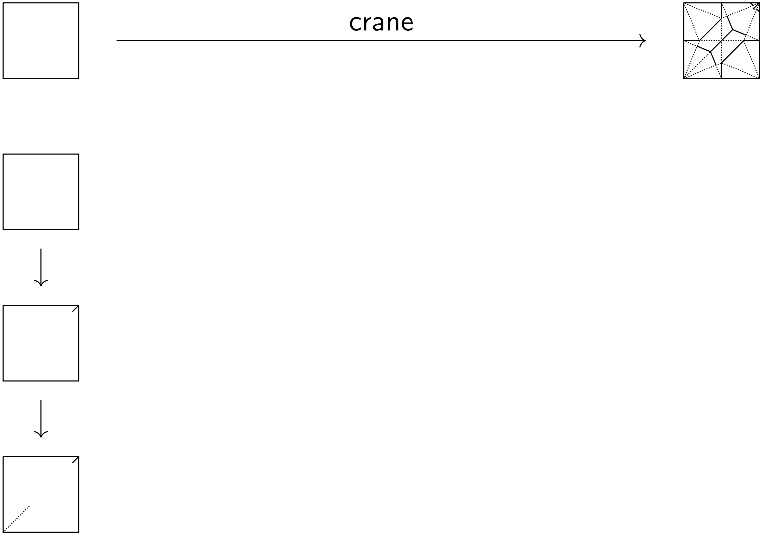

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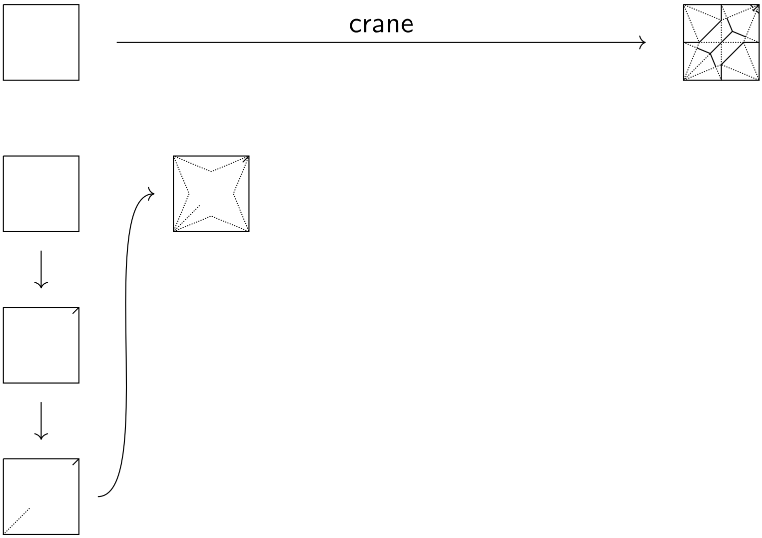

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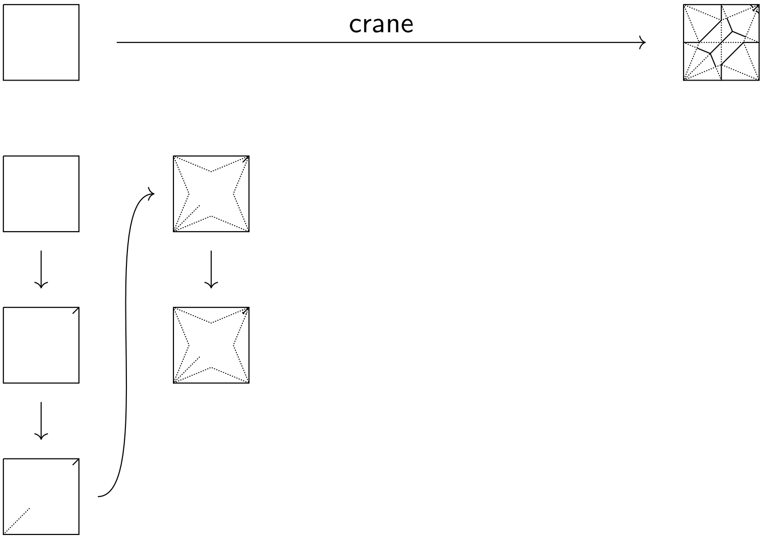

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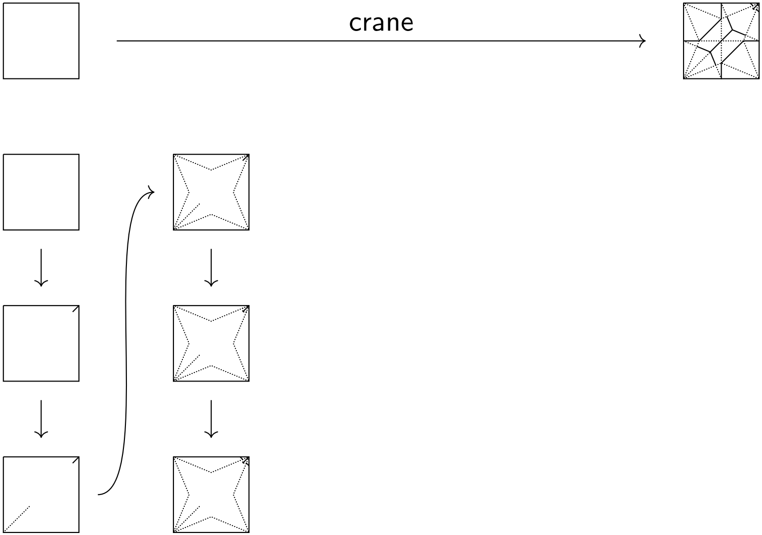

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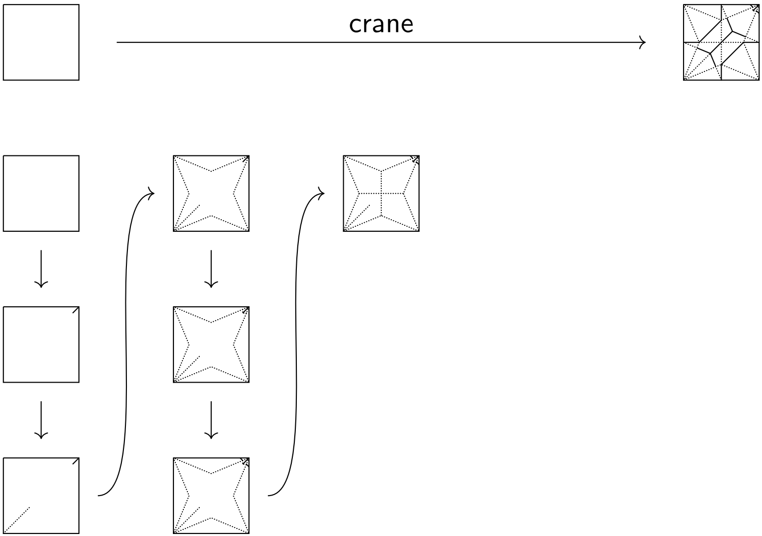

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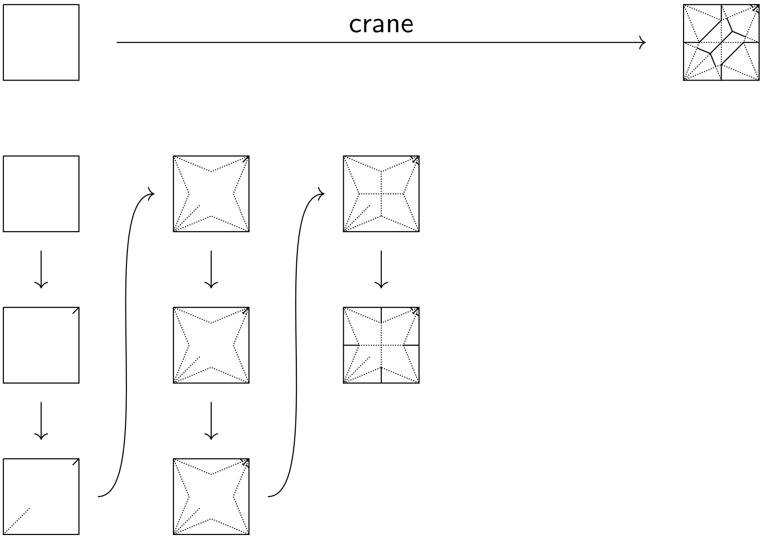

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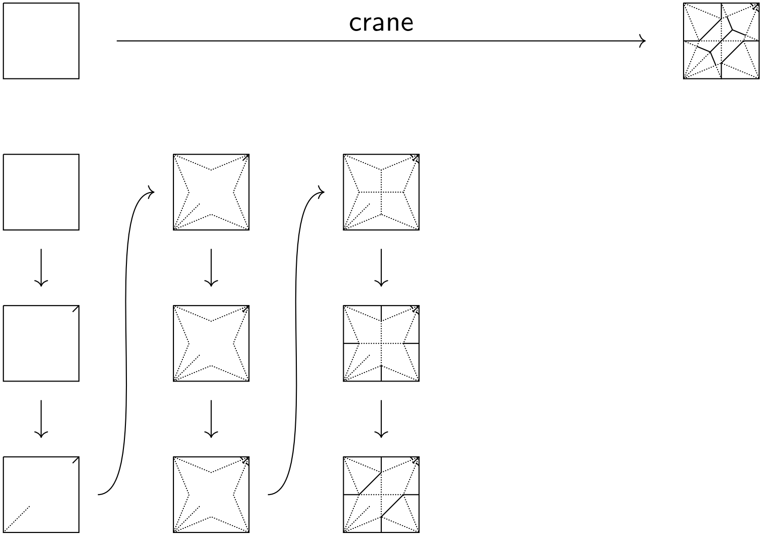

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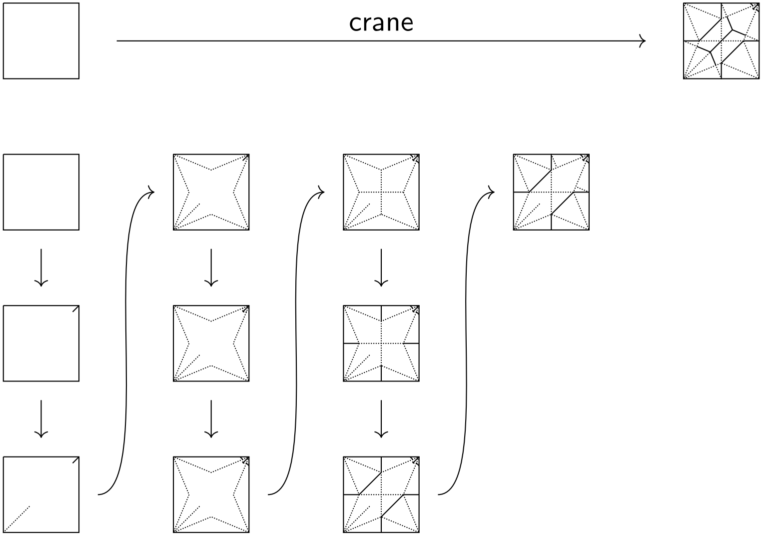

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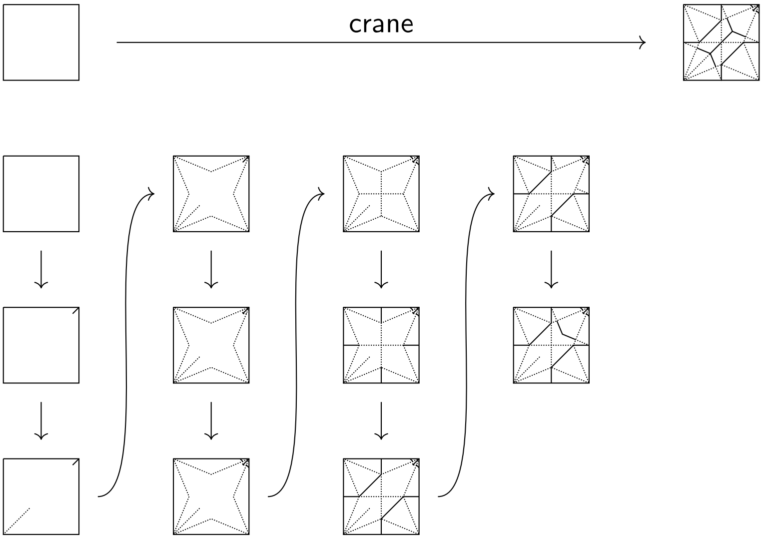

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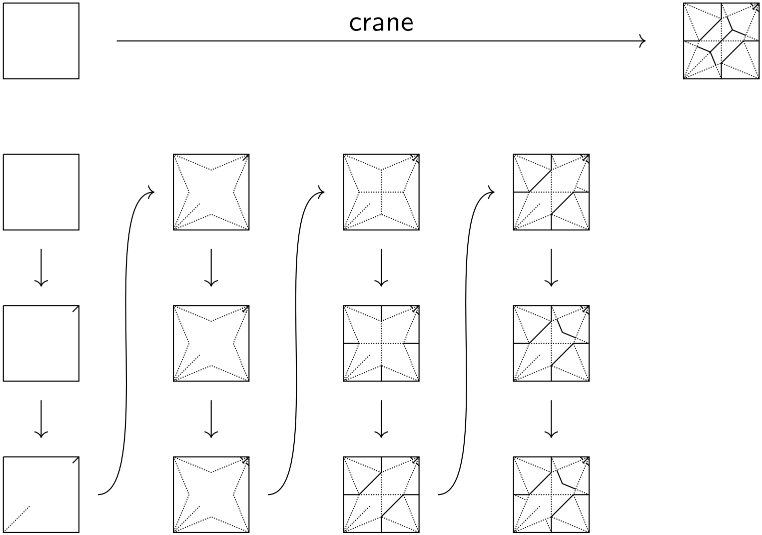

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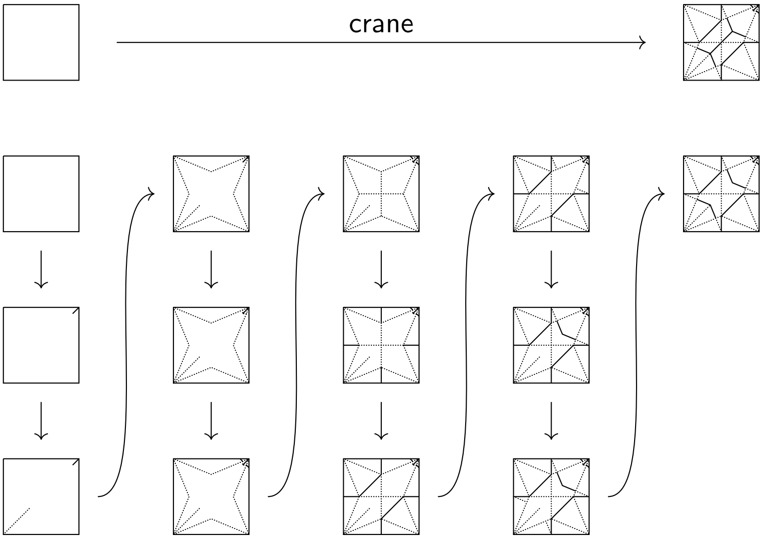

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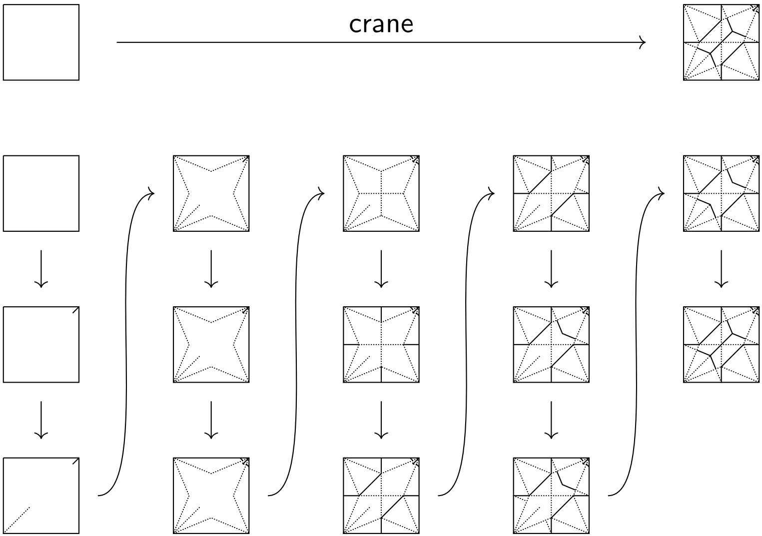
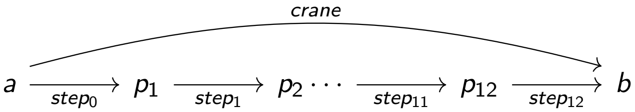


$crane = step_{12} step_{11} \cdots step_2 step_1 step_0$

This factorization increases the amount of information we have about *crane*.

We might write this as:



and say, "the diagram commutes," meaning, "paths whose source and target coincide are the same."

Of course, there may be many different factorizations...

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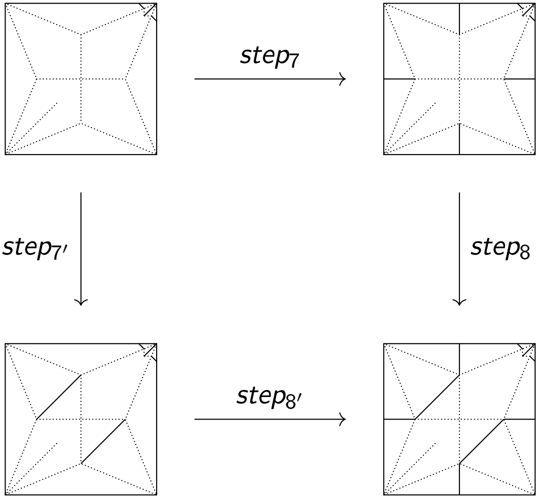

Why do the outer folds *before* the inner diagonal ones?

In other words, this diagram commutes.

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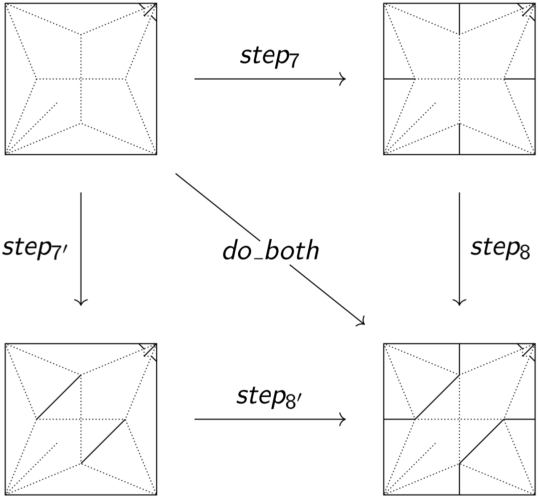

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Why not do both at once?

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Putting together or breaking apart maps—composing or factoring (resp.)—is natural in the context of describing an origami model

more than that it introduces the meaning of “natural” in category theory.

$$a \cdots \longrightarrow p_7 \xrightarrow{step_7} p_8 \xrightarrow{step_8} p_9 \cdots \longrightarrow b$$

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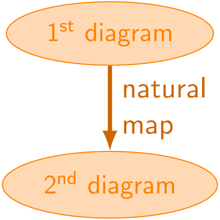

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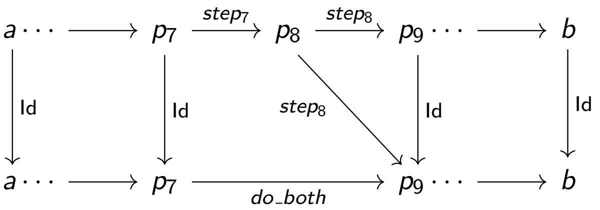

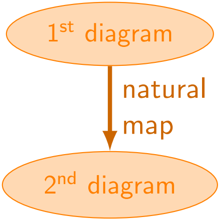

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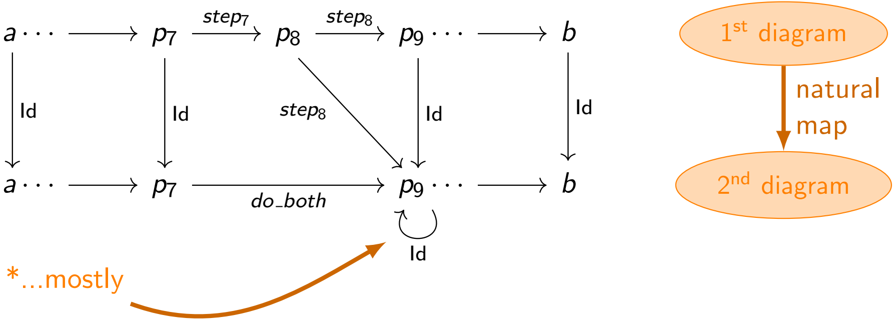

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Diagrams in categories of a given shape are functors from the category represented by that shape.

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naturality: For every morphism ($\circ \xrightarrow{\varphi} \bullet$) in the source category

$$\begin{array}{ccc} F\circ & \xrightarrow{F\varphi} & F\bullet \\ \eta_\circ\downarrow & & \downarrow\eta_\bullet \\ G\circ & \xrightarrow{G\varphi} & G\bullet \end{array} \quad \text{commutes.}$$

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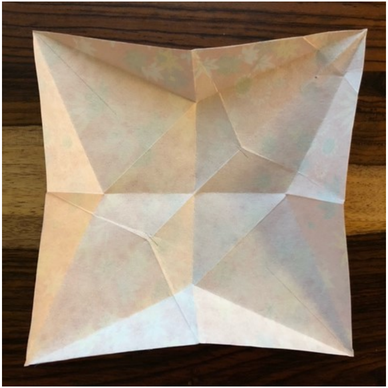
Natural Transformations: relationship-preserving maps between diagrams or maps of maps of categories.

Functors remove (or add) information

Where did the origami crane factorization come from?



It's a functor. A forgetful functor I made by removing information about the construction of this actual origami crane.

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Presheaves

In the other direction, a functor can act by "attaching" information to objects.

Definition-ish

A *presheaf* X is a functor $(\mathsf{C}^{\mathsf{op}} \xrightarrow{X} \mathsf{Set})$.

Presheaves form a category, denoted $[\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$, with

Objects: presheaves.

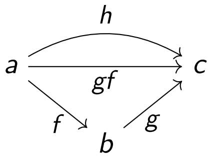
Morphisms: natural transformations.

Where a C-shaped diagram assigns *exactly one thing* to each object of C, a presheaf assigns a *set of things*.

The archetypal presheaf

The *representable presheaf at c*, denoted Yc or $\mathsf{C}(-, c)$, assigning to each $a \in \mathsf{C}$ the 'set'[1] of morphisms $(a \longrightarrow c)$.

Should think of it as "a copy of a for each way to get from a to c".



Yc is an indexing of the relationship "ends at c" and it propagates "backwards"[2]

[1] foundational issues are afoot.

[2] hence, the opposite category, C^{op}.

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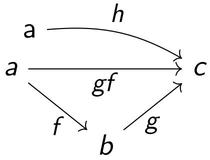

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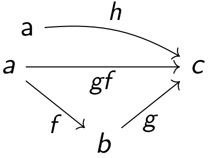

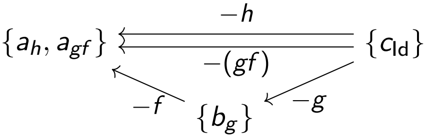

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The Yoneda Lemma (redux)

Natural transformations from the representable presheaf Yc to an arbitrary presheaf X are naturally determined by their value at $(c \xrightarrow{\mathrm{Id}_c} c)$.

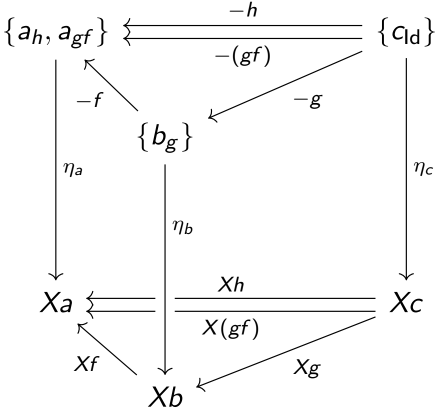
Lemma (Yoneda)

The following is a natural isomorphism in c and X:

$$\begin{array}{ccc} \eta & \longmapsto & \eta_c \,\mathrm{Id}_c \\ [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]\big(Yc, X\big) & \xleftrightarrow{\cong} & Xc \\ X - x & \longleftarrow\!\!\dashv & x \end{array} \tag{1.1}$$

Yc indexes the 'most commutative drawing of C' with respect to 'maps ending at c'.

Natural transformations from Yc



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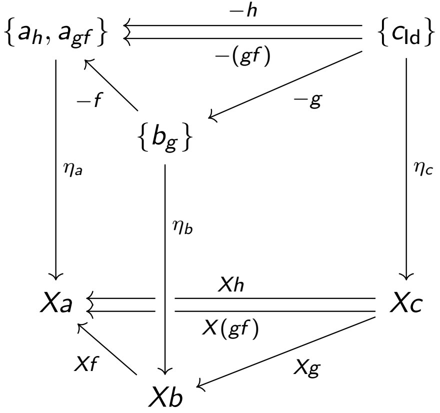
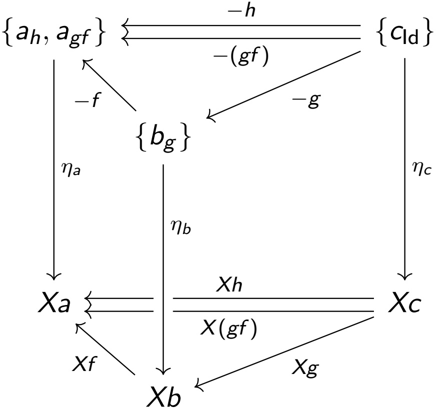


Q: What does η do to gf?

Natural transformations from Yc



Q: What does η do to gf?

A: By definition,

$$\begin{aligned} \eta_a(gf) &= Xf(\eta_b(g)) \\ &= Xf(Xg(\eta_c(\mathrm{Id}_c))) \\ &= X(gf)(\eta_c\, \mathrm{Id}_c) \end{aligned}$$

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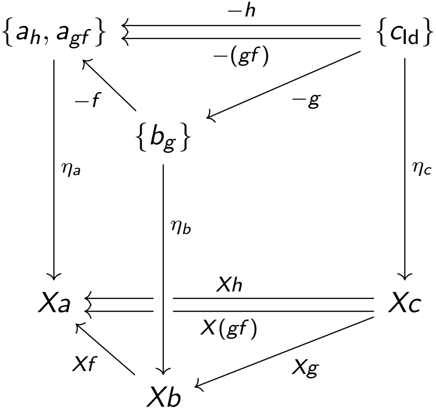

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But any map ending at c can be precomposed with the identity map at c.

Natural transformations preserve relationships encoded by commutative paths, by definition.

The presheaf Yc represents the the structure of 'paths to c' propagating naturally 'back from c' by precomposition.

Preserving that amounts to preserving *only* the behavior of the 'initial path to c,' namely the identity ($c \xrightarrow{\mathrm{Id}_c} c$).

Proof of Yoneda Lemma.

$$\begin{array}{ccc} \mathrm{C}(c,c) & \xrightarrow{\eta_c} & Xc \\ \Big\downarrow {\scriptstyle -f} & & \Big\downarrow {\scriptstyle Xf} \\ \mathrm{C}(b,c) & \xrightarrow{\eta_b} & Xb \end{array} \quad \text{therefore} \quad \begin{array}{ccc} \mathrm{Id}_c & \longmapsto & \eta_c\,\mathrm{Id}_c \\ \Big\downarrow & & \Big\downarrow \\ & & Xf(\eta_c\,\mathrm{Id}_c) \\ & & \| \\ f & \longmapsto & \eta_b(f) \end{array}$$

□